UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Friday, May 12, 2017

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 Suppose 10.0 g of water at a temperature of 100 °C is in an insulated cylinder equipped with a piston that maintains a constant pressure of p = 101.3 kPa. Enough heat is added to the water to vaporize it to steam at a temperature of 100 °C. The volume of the water is $V_{\text{water}} = 10.0\ \text{cm}^3$, and the volume of the steam is $V_{\text{steam}} = 16900\ \text{cm}^3$. What is the change in internal energy of the water? The latent heat of vaporization of water is $L_v = 2260\ \text{kJ/kg}$.

A2 A Carnot engine takes 3000 J of heat from a thermal reservoir that has a temperature of $T_{\text{H}} = 500$ K and discards heat to a thermal reservoir with a temperature $T_{\text{L}} = 325$ K. How much work does the Carnot engine do in this process?

A3 Suppose we start with 2.00 kg of water at a temperature of 20.0 °C and heat the water until it reaches a temperature of 80.0 °C. What is the change in entropy of the water?
The specific heat of water is $c = 4.19\ \text{kJ/(kg K)}$.

A4 The compression ratio of a diesel engine is 15.0 to 1; that is, air in a cylinder is compressed to $\frac{1}{15.0}$ of its initial volume.

a. If the initial pressure is 1.01×10^5 Pa, and the initial temperature is 27°C (300 K), find the final pressure and the temperature after adiabatic compression.

b. How much work does the gas do during the compression if the initial volume of the cylinder is 1.00 liter?

Use the values $C_V = 20.8\ \text{J/(mol} \cdot \text{K)}$ and $\gamma = 1.400$ for air.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

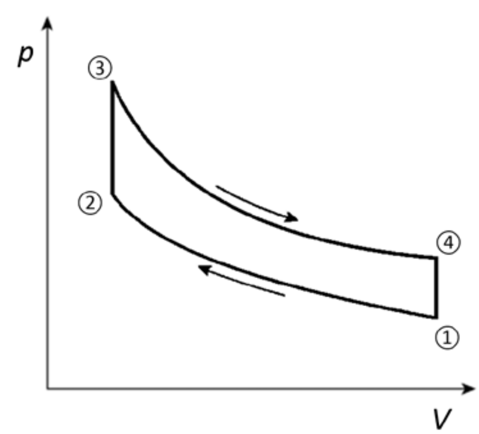
B1 I show you four unusual six-sided dice. They are unusual because they do not have the numbers 1 through 6 on their six faces. Instead, here's what the dice look like (what I'm going to do is list the six numbers on the faces, in increasing order):

- Die A: 0, 0, 4, 4, 4, 4
- Die B: 3, 3, 3, 3, 3, 3
- Die C: 2, 2, 2, 2, 6, 6
- Die D: 1, 1, 1, 5, 5, 5

I play the following game with you: you choose one of the dice, and then, knowing which die you chose, I choose another. You are always the first to choose. Having made our choices, we both roll our respective dice. You win the game if you roll a higher number than me (notice that ties are impossible).

Having seen which die you took, I always take, from the remaining three dice, the one that maximizes *my* probability of winning. Given my strategy, which of the dice should you choose (you always choose first) in order to maximize *your* probability of winning this game, and what is your probability to win with that choice?

B2 The Otto cycle, used in modern internal combustion engines, consists of two adiabatic processes and two constant-volume processes. Given that the volume of the fuel-air mixture at points (1) and (4) is V_1 and the volume of the mixture at points (2) and (3) is V_2, find the efficiency of the Otto cycle (i.e. net work done over the heat gained per cycle).



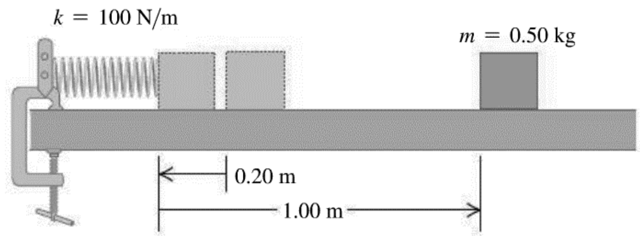
B3 For the internal energy $U(V,T)$ compute $(\partial U/\partial V)_p$ as a function of observable quantities only: pressure p, temperature T, volume V, isothermal compressibility κ, and thermal expansion coefficient α (refer to the formula sheet for the definitions of κ and α).

B4 Consider a system of N Ising spins. Each spin has two states, up or down, with energy $+J$ or $-J$. The spins do not interact. E/N is the total energy per spin as $N \to \infty$.

a. What is the maximum value of E/N, when no thermodynamic equilibrium is assumed?
b. What is maximum value in thermodynamic equilibrium at $T > 0$?

Mechanics Group A - Answer only two Group A questions

A1 A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m. When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The spring constant is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?



A2

A ball of mass 1 kg is falling vertically down. Assuming quadratic air resistance, $F = cv^2$ (v is the ball's speed) with $c = 0.01$ kg/m,

(a) Find the terminal speed;

(b) Assuming that the ball is moving with the terminal speed, find the energy dissipated per second.

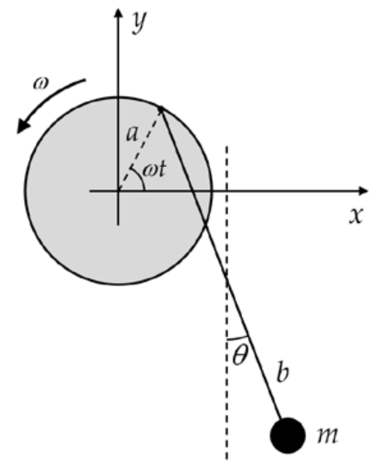
A3 Gravitation.

A spherical shell filled with a fluid of mass M has the inner radius R_1 and the outer radius R_2. A particle of mass m is placed at the distance r from the center. Find the force $F(r)$ on the particle at (a) $r < R_1$, (b) $R_1 < r < R_2$ and (c) $r > R_2$, and sketch the function $F(r)$

A4 A positively charged object (mass m_1) approaches another positively charged object (mass m_2) at rest. Long after the collision, both objects travel in the same direction. What are the final velocities v_1 and v_2 of m_1 and m_2, respectively?

Mechanics Group B - Answer only two Group B questions

B1 The point of support of a simple pendulum of length b is attached to the rim of a massless wheel of radius a that rotates with constant angular velocity ω.

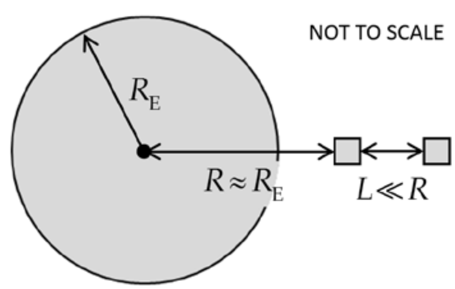


a. Obtain the expression for the Cartesian components of the velocity of the mass m.
b. Obtain the Lagrangian.
c. Obtain the equation of motion for θ (obtain the angular acceleration for the angle θ shown in the figure).

B2 Central-force motion.

a. Find the central-force field $F(r)$ that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants. You may use the equation of the orbit, $\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{mr^2}{L^2}F(r)$, where $L = mr^2(d\theta/dt) = \text{constant}$.
b. Find $\theta(t)$.

B3 Two satellites are connected by a rope. Both satellites orbit the Earth with the same angular velocity. The distance from the center of the Earth to the first satellite is R, which is approximately equal to R_E, the radius of the Earth. The length of the rope is L, with $L \ll R$. Both satellites are on a ray projecting from Earth's center. In the limit $L \ll R$, what is the tension in the rope?



B4 A chain has mass M and length L. It is suspended vertically with its lower end at a distance L from a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length x of the chain is on the scale? Neglect the size of the individual links.

Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381 \times 10^{-23}$ J/K

elementary charge .... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

Avogadro constant .... $N_A = 6.022 \times 10^{23}$ mol^{-1}

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109 \times 10^{-31}$ kg

electron rest energy...... 511.0 keV

Compton wavelength .. $\lambda_C = h / m_{el}c = 2.426$ pm

proton mass .... $m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836 m_{el}$

1 bohr............................ $a_0 = \hbar^2 / ke^2 m_{el} = 0.5292$ Å

1 hartree (= 2 rydberg) ... $E_h = \hbar^2 / m_{el} a_0^{\,2} = 27.21$ eV

gravitational constant ... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

hc .... $hc = 1240$ eV·nm

Equations That May Be Helpful

POWER SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \left(|x| \le 1,\ x \ne -1\right)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \left(|x| < 1\right)$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \dfrac{|W|}{|Q_h|}$.

For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \dfrac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \le 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const.}$

$$dU = TdS - pdV \qquad dH = d(U + pV)$$
$$dF = d(U - TS) \qquad dG = d(U + pV - TS)$$

$$H = U + pV \qquad F = U - TS \qquad G = F + pV \qquad \Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Triple product: $\left(\frac{\partial X}{\partial Y}\right)_Z \cdot \left(\frac{\partial Y}{\partial Z}\right)_X \cdot \left(\frac{\partial Z}{\partial X}\right)_Y = -1$

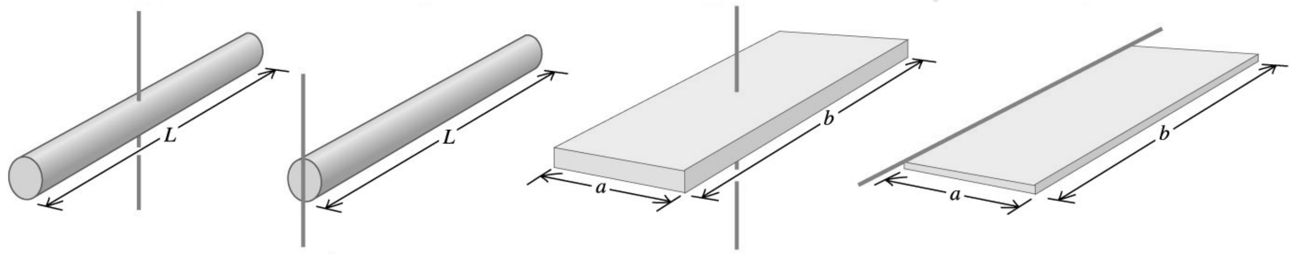
specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

MECHANICS

Gravitational acceleration at surface of Earth: $g = 9.81\ \text{m/s}^2$

Gauss's Law for gravity: $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{encl.}}$

Moments of Inertia of Various Bodies

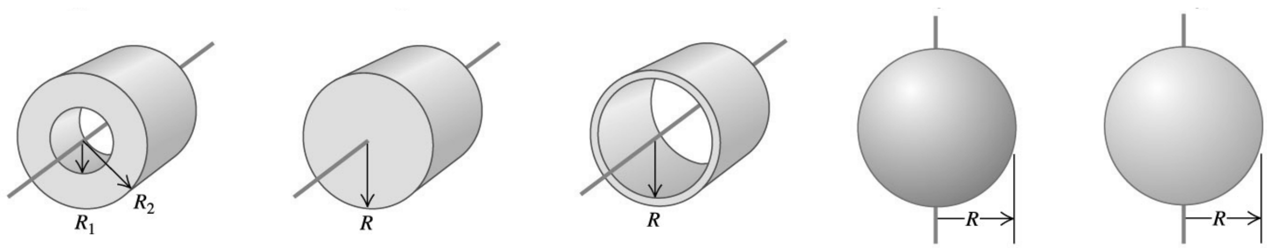


Slender rod, axis through center: $I = \frac{1}{12}ML^2$

Slender rod, axis through one end: $I = \frac{1}{3}ML^2$

Rectangular plate, axis through center: $I = \frac{1}{12}M(a^2 + b^2)$

Thin rectangular plate, axis along edge: $I = \frac{1}{3}Ma^2$



Hollow cylinder: $I = \frac{1}{2}M(R_1^2 + R_2^2)$

Solid cylinder: $I = \frac{1}{2}MR^2$

Thin-walled hollow cylinder: $I = MR^2$

Solid sphere: $I = \frac{2}{5}MR^2$

Thin-walled hollow sphere: $I = \frac{2}{3}MR^2$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

INTEGRALS

$f(x)$	$\int_0^\infty f(x)\,dx$
e^{-ax^2}	$\dfrac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	$\dfrac{1}{2a}$
$x^2e^{-ax^2}$	$\dfrac{\sqrt{\pi}}{4a^{3/2}}$
$x^3e^{-ax^2}$	$\dfrac{1}{2a^2}$
$x^4e^{-ax^2}$	$\dfrac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5e^{-ax^2}$	$\dfrac{1}{a^3}$
$x^6e^{-ax^2}$	$\dfrac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}dy = n!a^{n+1}$$

$$\int x^{-2}\ln(x)\,dx = -\frac{\ln(x)}{x} - \frac{1}{x}$$

$$\int x^{-1}\ln(x)\,dx = \tfrac{1}{2}\ln^2(x)$$

$$\int \ln(x)\,dx = x\ln(x) - x$$

$$\int x\ln(x)\,dx = \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2$$

$$\int x^2\ln(x)\,dx = \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{9}x^3$$

$$\int \ln(1+x)dx = (x+1)\big(\ln(x+1)-1\big)+C$$

$$\int \frac{r^3dr}{(x^2+r^2)^{3/2}} = (r^2+x^2)^{1/2} + \frac{x^2}{(r^2+x^2)^{1/2}}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\ln\left(a^2+x^2\right)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right) = -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2$$