UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
May 16, 2014

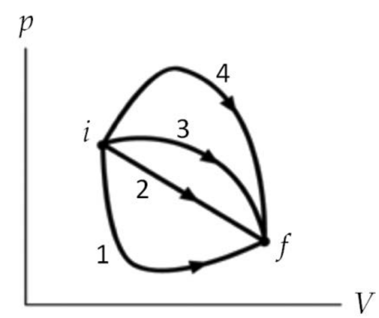
This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Mechanics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A - Answer only two Group A questions

A1 Two editors work independently and examine a manuscript that contains N typos. They are both equally well-trained and spot typos with probability equal to p. What is the expected number of typos that neither of the editors will spot?

A2 A gas can be taken from the initial state "i" to the final state "f" along the four different paths shown on a p-V diagram in the figure. Which of the paths corresponds to the highest amount of heat absorbed by the gas? Explain your reasoning.

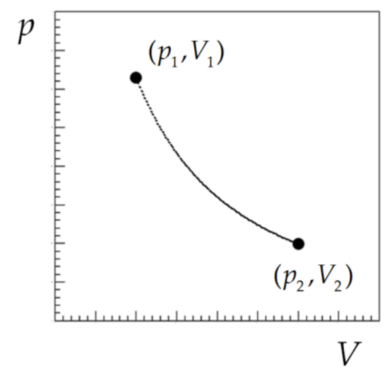


A3 A reversible Carnot cycle is used as a heat pump to heat a room at temperature 20°C, while outside air at 0°C serves as a heat source. How much power must be supplied in order to provide the heat to the room at 1 kW?

A4 In a first experiment, a certain gas expands into vacuum after an internal partition in an enclosing thermally insulating vessel is punctured. It was found that the temperature of the gas has decreased. In a second experiment, the gas slowly expands isothermally as it absorbs heat Q and does work W. Based on the result of the first experiment, predict whether Q is greater, smaller, or equal to W.

Thermodynamics and Statistical Mechanics Group B - Answer only two Group B questions

B1 Consider a mole of ideal gas taken through a process with the initial state at (p_1,V_1,T_1) and the final state at (p_2,V_2,T_2).

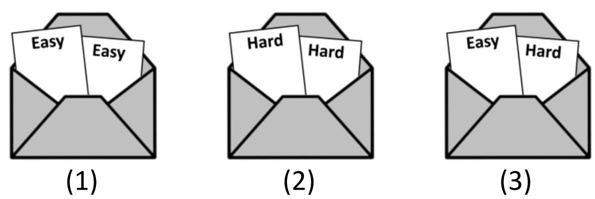


a. Show that for any two states 1 and 2, the following holds:

$$S_2 - S_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right).$$

b. Show that if the states 1 and 2 lie on an adiabatic curve, the difference $S_2 - S_1$ vanishes. Assume $C_p = C_V + R$, but do not assume any particular value of C_V.

B2 In a statistics course you have learned to solve an assortment of easy and hard problems. For the final exam the professor prepares closed envelopes of 3 types: (1) with two easy problems, (2) with two hard problems, and (3) with one easy and one hard problem. There are an equal number of envelopes of each type, and you pull a closed envelope at random. After opening your envelope, you examine *one* of the problems and it turns out to be hard. What is the probability that the other problem in your envelope is also hard?



B3 A 3 kg bronze brick heated to 90°C is placed in a thermally insulated vessel with 600 g of ice, which is initially at $-15°C$. The whole system is then allowed to reach equilibrium.

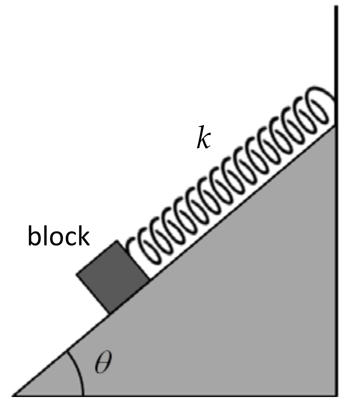
a. What is the final temperature of the system?
b. What was the entropy change of the whole system in this process?

Reference data: Specific heat of water: 4186 J / (kg·K); ice: 2050 J / (kg·K); bronze: 435 J / (kg·K).
It takes 334 joules to melt 1gram of ice.

B4 Show that if a system's initial and final temperature and pressure are that of a bath that is both isothermal and isobaric, then the maximum useful work W done by this system is $W_{\max} = -\Delta G$, that is, the negative change in the Gibbs free energy of the system.

Mechanics Group A - Answer only two Group A questions

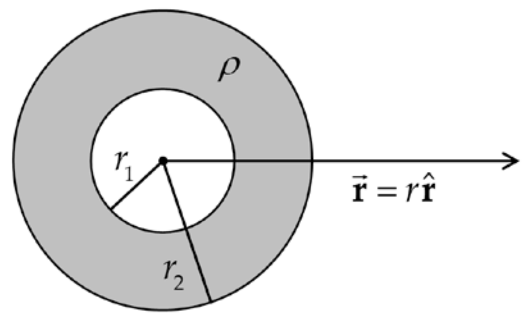
A1 In the figure, a block weighing 14.0 N, which can slide without friction on an incline at angle $\theta = 40°$, is connected to the top of the incline by a massless spring of un-stretched length 0.45 m and spring constant $k = 120$ N/m.



a. How far from the top of the incline is the block's equilibrium point?

b. If the block is pulled slightly down the incline and released, what is the period of the resulting oscillations?

A2 Consider a mass shell of finite thickness with inner radius r_1 and outer radius r_2, as shown in the figure. The shell has uniform volume mass density ρ.



Calculate the gravitational force on a mass m …

a. … at $r < r_1$

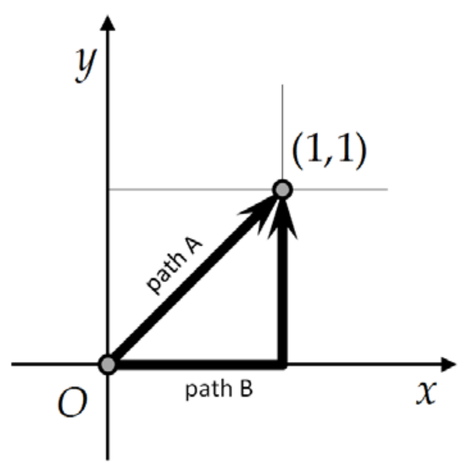
b. … at $r_1 < r < r_2$

c. … at $r > r_2$

A3 *This problem is 2-dimensional.* Consider the two force functions

$$\mathbf{F}_1 = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \text{ , and}$$
$$\mathbf{F}_2 = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} \text{ .}$$

The adjacent diagram shows two different paths (A and B) from the origin to the point (1,1).



a. Calculate $\int_{\text{path A}} \mathbf{F}_1 \cdot d\mathbf{r}$ and $\int_{\text{path B}} \mathbf{F}_1 \cdot d\mathbf{r}$.

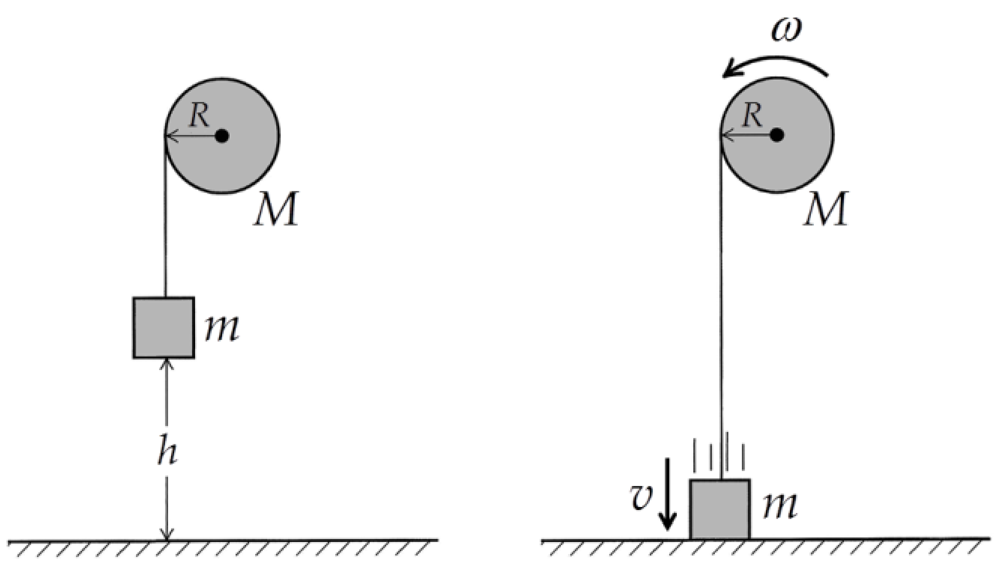
Is $\mathbf{F}_1$ conservative? Explain.

b. Calculate $\int_{\text{path A}} \mathbf{F}_2 \cdot d\mathbf{r}$ and $\int_{\text{path B}} \mathbf{F}_2 \cdot d\mathbf{r}$.

Is $\mathbf{F}_2$ conservative? Explain.

A4 *See diagram.* We wrap a massless, flexible cable around a solid uniform cylinder with mass M and radius R. The cylinder rotates without friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the object with no initial velocity at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping, turning the cylinder.

Find the speed of the falling block and the angular speed of the cylinder just as the block strikes the floor.



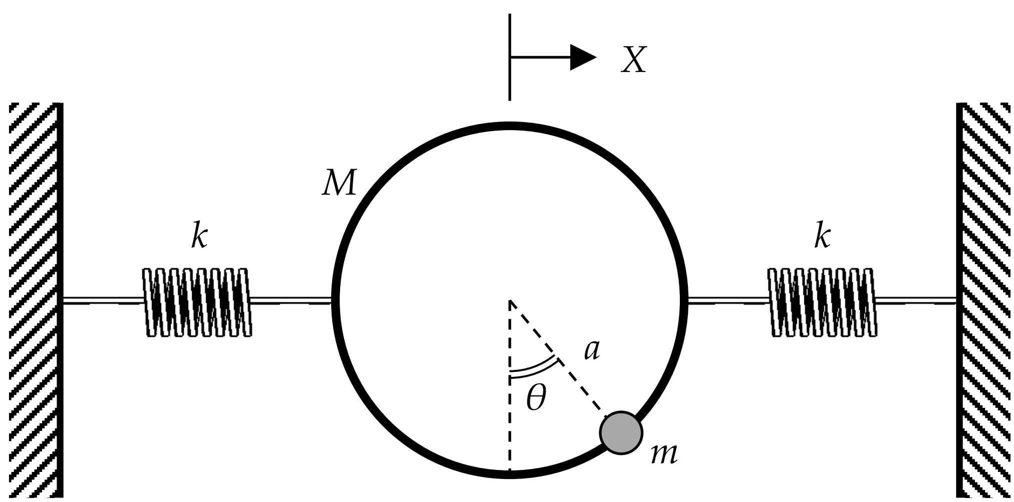
Cylinder and block at rest — Block about to hit ground

Mechanics Group B - Answer only two Group B questions

B1 Consider a particle subject to an attractive central force of magnitude $f(r)$, where r is the distance between the particle and the center of the force. Find the explicit form $f(r)$ if all circular orbits are to have identical areal velocities A. (Note: areal velocity, a.k.a. sectorial velocity, is the rate at which area is swept out by a particle as it moves along the orbit).

B2 A thin, uniform stick of length ℓ is initially in unstable equilibrium with its bottom end resting on a frictionless table. A small disturbance makes the stick fall. Find the velocity of the center of mass when it reaches the table.

B3 A point mass m slides frictionlessly, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as depicted in the figure. All motion is within the plane of the figure. Except for the point mass, there is no motion in the *vertical* direction: the ring is constrained without friction to move horizontally.



Choose as generalized coordinates the horizontal displacement X of the center of the ring with respect to equilibrium, and the angle θ a radius to the mass m makes with respect to the vertical (see fig. 1). You may assume that at $X=0$ the springs are both unstretched.

a. Find the Lagrangian $L(X,\theta,\dot{X},\dot{\theta},t)$
b. Find the momenta conjugate to X and θ.
c. Derive the equations of motion (but don't solve them).

B4 A particle of mass m is performing one-dimensional motion subject to the force function $F(x) = -F_0 \sin(cx)$. At some instant the particle's position is $x = 0$ and its velocity $v = v_0$.

a. Find the potential energy as a function of x and sketch it.
b. Find the velocity as a function of x.
c. Find the condition on the initial velocity v_0 for which the motion is periodic ($v_0 <$?).
d. Suppose that v_0 is so small that the periodic motion can be treated as a harmonic oscillation. Find the period of this oscillation.

Physical Constants

speed of light.................. $c = 2.998\times 10^{8}$ m/s

Planck's constant .... $h = 6.626\times 10^{-34}$ J·s

Planck's constant / 2π.... $\hbar = 1.055\times 10^{-34}$ J·s

Boltzmann constant ...... $k_B = 1.381\times 10^{-23}$ J/K

elementary charge .... $e = 1.602\times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854\times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257\times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988\times 10^{9}$ m/F

electron mass .... $m_{el} = 9.109\times 10^{-31}$ kg

proton mass .... $m_p = 1.673\times 10^{-27}$ kg

1 bohr............................ $a_0 = 0.5292$ Å

1 hartree .... $E_h = 27.21$ eV

Avogrado constant ..... $N_A = 6.022\times 10^{23}$ mol^{-1}

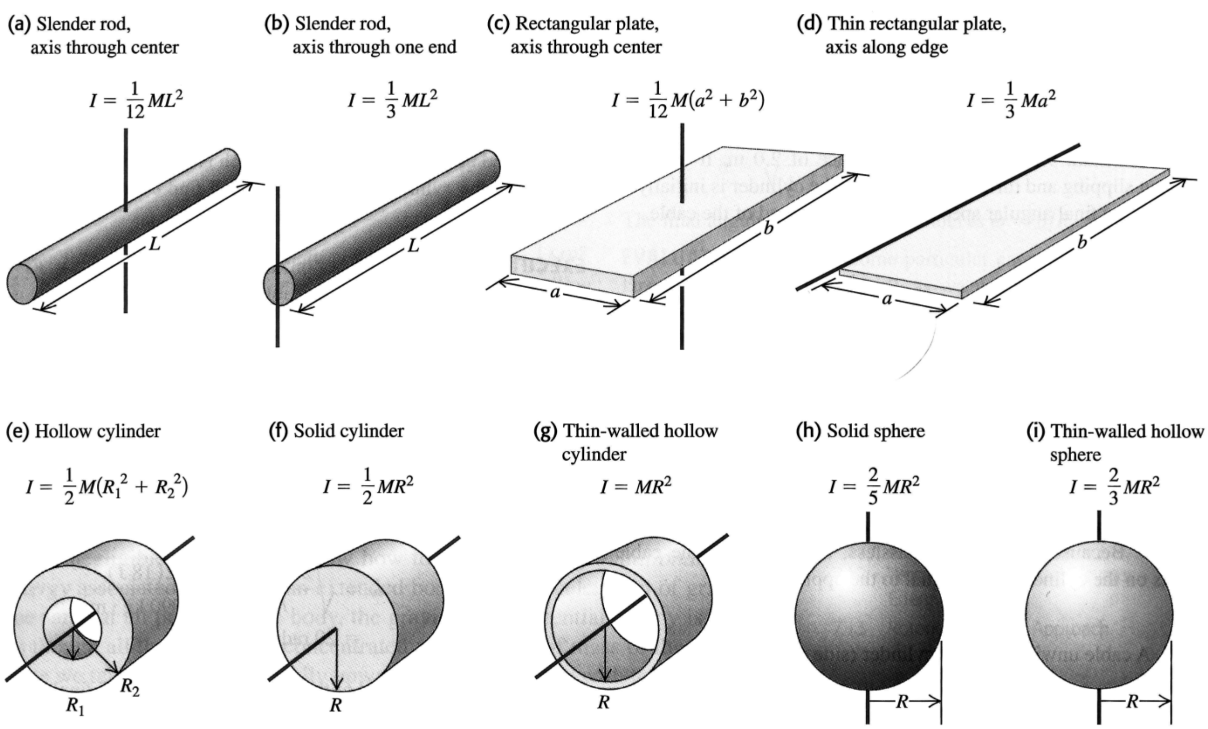
gravitational constant.. $G = 6.674\times 10^{-11}$ m^3/ kg s^2

EQUATIONS THAT MAY BE HELPFUL

MECHANICS:

Gravitational acceleration on Earth: $g = 9.81$ m/s^2

Moments of Inertia of Various Bodies



THERMODYNAMICS

General efficiency η of a heat engine producing work $|W|$ while taking in heat $|Q_h|$ is $\eta = \frac{|W|}{|Q_h|}$

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For a Carnot cycle operating as a heat engine between reservoirs at T_h and at T_c the efficiency becomes $\eta_C = \frac{T_h - T_c}{T_h}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$, which becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

For adiabatic processes in an ideal gas with constant heat capacity, $PV^\gamma = \text{constant}$.

$$\frac{dP}{dT} = \frac{\lambda}{T\Delta V}$$

specific heat of water: 4186 J/(kg·K)
latent heat of ice melting: 334 J/g

$$H = E + PV \qquad F = E - TS \qquad G = F + PV \qquad \Omega = F - \mu N$$

$$dE = TdS - PdV + \mu dN \qquad dS = dE/T + PdV/T - \mu dN/T \qquad dH = TdS + VdP + \mu dN$$

$$dF = -SdT - PdV + \mu dN \qquad dG = -SdT + VdP + \mu dN \qquad d\Omega = -SdT - PdV - Nd\mu$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \qquad C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

INTEGRALS

$f(x)$		$\int_0^\infty f(x)\,dx$
e^{-ax^2}	....	$\frac{\sqrt{\pi}}{2\sqrt{a}}$
xe^{-ax^2}	....	$\frac{1}{2a}$
$x^2 e^{-ax^2}$	....	$\frac{\sqrt{\pi}}{4a^{3/2}}$
$x^3 e^{-ax^2}$	....	$\frac{1}{2a^2}$
$x^4 e^{-ax^2}$	....	$\frac{3\sqrt{\pi}}{8a^{5/2}}$
$x^5 e^{-ax^2}$	....	$\frac{1}{a^3}$
$x^6 e^{-ax^2}$	....	$\frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\int_0^\infty \frac{1}{1+ay^2}\,dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a}\,dy = n!\,a^{n+1}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2$$