UNL - Department of Physics and Astronomy

# Preliminary Examination - Day 1

# May 15, 2014

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

**WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY**

**Quantum Mechanics Group A** - Answer only two Group A questions

**A1** A particle is moving in a central potential. Its angular momentum quantum numbers are $\ell, m_z$. Find the expectation value of the operator $(\hat{L}_x)^2$.

**A2** The photoelectric threshold of tungsten is 230 nm. Determine the energy of the electrons ejected from the surface by ultraviolet light of wavelength 190 nm. *Info: hc = 1240 eV·nm*

**A3** A free particle of mass $m$ moves in one dimension in the range $x > 0$ with energy $E$. There is a rigid, infinitely high wall at $x = 0$.

*a.* Write down the solutions of the stationary and time-dependent Schrödinger equations for this problem.

*b.* Is this solution a momentum eigenstate? A parity eigenstate? Explain.

**A4** A particle is moving in one dimension in the potential $V(x) = F|x|$, where $F$ is a positive constant.

*a.* Sketch this potential and indicate the turning points for a given energy $E$.

*b.* Sketch an even solution of the corresponding Schrödinger equation. Your sketch should make clear the behavior of the wave function in different regions with respect to the turning points.

**Quantum Mechanics Group B** - Answer only two Group B questions

**B1** Consider a stationary electron in an external uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

*a.* Write down the electronic Hamiltonian and find its eigenvalues.

The electron spin along the $x$ axis is measured at $t = 0$ and found to be $\frac{1}{2}\hbar$.

*b.* What is the probability that the electron spin along the $x$ axis is $\frac{1}{2}\hbar$ at time $t$?

*c.* What is the probability that the electron spin along the $x$ axis is $-\frac{1}{2}\hbar$ at time $t$?

*d.* What is the probability that the electron spin along the $z$ axis is $\frac{1}{2}\hbar$ at time $t$?

*Note: the electron magnetic moment operator is* $-\frac{e}{m}\hat{\mathbf{S}}$, *where* $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ *is the electron spin operator.*

**B2** A particle of mass $m$ is moving in the harmonic potential $\frac{1}{2}m\omega^2 x^2$. At $t = 0$ the wave function is given by

$$\Psi(x, t = 0) = \psi_0(x) + \psi_1(x),$$

where $\psi_0(x), \psi_1(x)$ are the lowest two energy eigenstates.

*a.* Find $\Psi(x, t)$ for $t > 0$.

*b.* Find the expectation value of $x$ as a function of time.

*c.* Using the Heisenberg equation of motion, find the expectation value of the velocity $v$ as a function of time.

**B3** The radial wave function of the hydrogen atom in its ground state is

$$R(r) = 2\frac{e^{-r/a_0}}{a_0^{3/2}},$$

where $a_0$ is the Bohr radius. Find:

*a.* the average value of $r$;

*b.* the most probable value of $r$;

*c.* the largest value of $r$ that is classically allowed; and

*d.* the probability to find the electron in the classically forbidden region.

**B4** A particle of mass $m$ is moving in one-dimensional rectangular well. The potential is given by

$$V(x) = \begin{cases} -V_0 & \text{for } |x| < L \\ 0 & \text{for } |x| > L \end{cases}$$

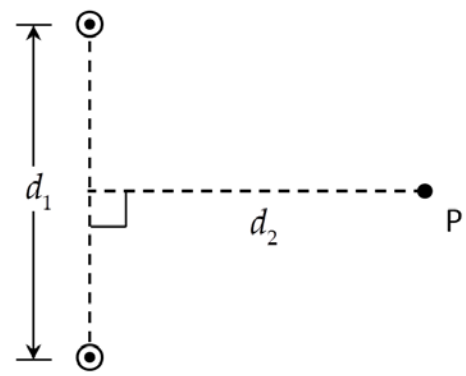
*a.* Consider a parity-even (symmetric) state and derive the equation allowing you to determine the energy eigenvalues. (You don't have to solve this equation).

*b.* Consider the ground state in the limit $V_0 \to \infty$. What do you expect for the energy eigenvalue? Show that this result can be obtained from the equation derived in part *a*.

**Electrodynamics Group A** - Answer only two Group A questions

**A1** The figure shows two very long, straight wires (in cross section) that each carry a current 4.0 A directly out of the page. The distance $d_1 = 6.0 \text{ m}$ and distance $d_2 = 4.0 \text{ m}$.

What is the magnitude of the magnetic field at point P?




**A2** In the manufacture of aluminum (Al), an electric current passes through a melt containing Al ions (mostly $Al^{3+}$) and $O^{2-}$ ions. The aluminum ions move to the negative electrode (the "cathode") onto which they are deposited, and the oxygen ions move to the positive electrode (the "anode"). The melt remains electrically neutral.

Estimate the average current through the melt so that a metric ton ($10^3$ kg) of aluminum per day is deposited at the cathode. The atomic mass of aluminum is 27.

**A3** A particle detector comprises a fine wire of radius 10 μm on the axis of a long conducting cylinder of radius 1 cm. The cylinder is held at ground and the wire at +3.5 kV. The tube is filled with gas atoms that are ionized if struck by free electrons of kinetic energy greater than 4 eV. The distance between collisions for free electrons in the gas is 0.1 μm. Assume that such a collision results in two free electrons, each initially at rest.

Estimate

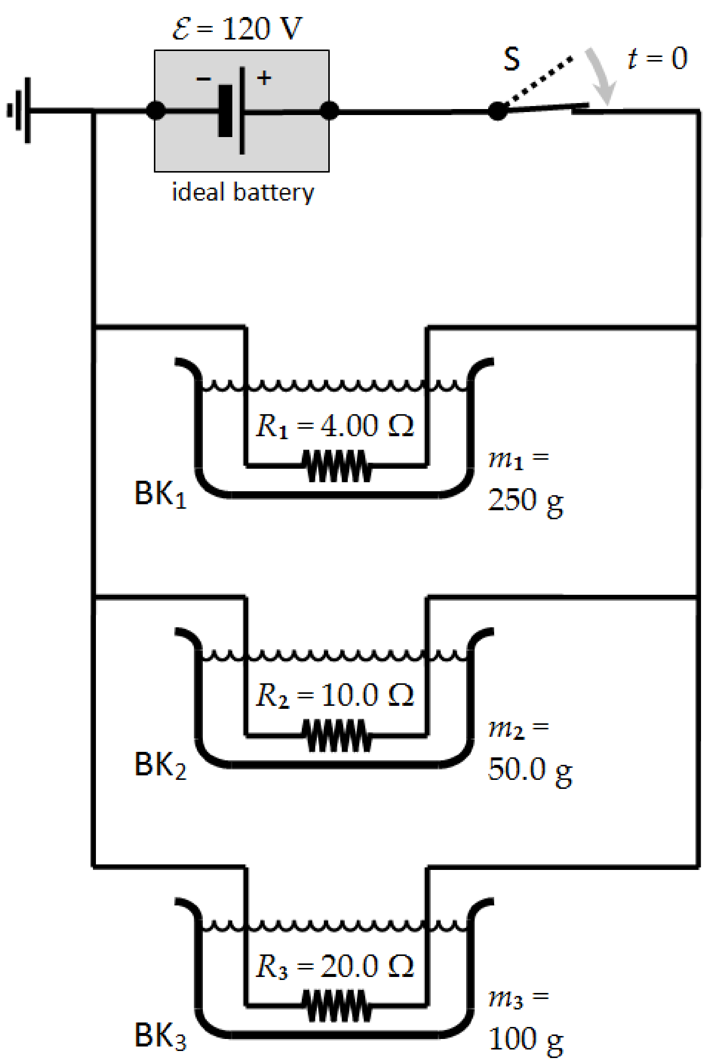
*a.* the distance from the inner wire's surface at which a free electron will cause ionization when it collides with an atom; and

*b.* the number of electrons arriving at the wire initiated by one electron in the detector.

**A4** An ideal battery containing a source of emf with $\mathcal{E} = 120$ V, is used to heat up three beakers ($BK_1$, $BK_2$, and $BK_3$) of water. This is done by immersing a resistor in each beaker, and connecting these resistors to the battery as shown in the diagram. The masses of the water quantities in the three beakers are $m_1 = 250$ g, $m_2 = 50.0$ g, and $m_3 = 100$ g. All water initially has a temperature of 20.0°C. At time $t = 0$, the switch S is closed. Assume the resistance of the resistors does not depend on their temperature. After some time, the water in one of the beakers will start to boil.

Identify which beaker boils first. Calculate the water temperature in the other two beakers at the moment the water in that first beaker starts to boil.

*Info: The specific heat capacity of water is* $C = 4.18$ J g$^{-1}$ °C$^{-1}$.

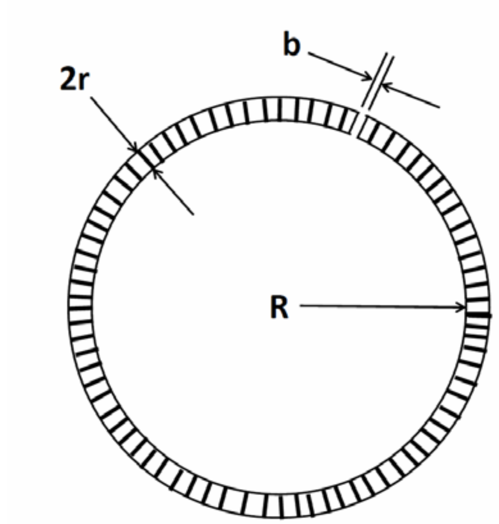


**Electrodynamics Group B** - Answer only two Group B questions

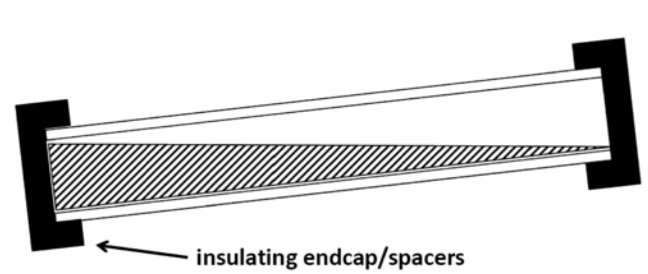
**B1** A charge $Q$ is uniformly distributed over a sphere of radius $R$ (uniform volume charge density). Taking the electrostatic potential to be zero infinitely far away from this sphere, what is the sphere's average electrostatic potential?

**B2** A ring of mean radius $R$ and circular cross section $\pi r^2$, with $r \ll R$, is made from a material with high relative permeability $\mu_r$. The ring is tightly and uniformly wound with $N$ turns of wire which carries a current $I$.

*a.* Estimate the magnitude of the $B$-field inside the material of the ring.
*b.* If a gap of length $b$ is now cut in the ring, with $b \ll r$, what is the approximate magnetic field $B$ in the gap?




**B3** A charged and isolated parallel-plate capacitor is half-filled with an insulating liquid of dielectric constant $\varepsilon_r$. If $V_h$ and $V_v$ are the potential differences between the plates when they are horizontal and vertical, respectively, what is $V_h / V_v$?




**B4** A flexible, conducting wire of radius $a$, length $L$, and Young's modulus $Y$ is connected to two electrical terminals a distance $L$ apart so that it is supported without tension. Ignore the wire's weight. The wire is immersed in a magnetic field $B$ perpendicular to its length.

*a.* If a small current $I$ is now passed through the wire, what is the tension in the wire?
*b.* What is its maximum transverse displacement perpendicular to the $B$-field direction?
*c.* Estimate the two quantities above if $B = 1\ \mathrm{T}$, $I = 1\ \mathrm{A}$, $L = 1\ \mathrm{m}$, $a = 1\ \mathrm{mm}$, and $Y = 10^{11}\ \mathrm{N/m^2}$.

## Physical Constants

speed of light.................. $c = 2.998 \times 10^{8}$ m/s

Planck's constant ........... $h = 6.626 \times 10^{-34}$ J·s

Planck's constant / $2\pi$.... $\hbar = 1.055 \times 10^{-34}$ J·s

Boltzmann constant ...... $k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J/K

elementary charge ......... $e = 1.602 \times 10^{-19}$ C

electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

magnetic permeability ... $\mu_0 = 1.257 \times 10^{-6}$ H/m

molar gas constant.......... $R = 8.314$ J / mol·K

electrostatic constant ... $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ m/F

electron mass ............... $m_{\mathrm{el}} = 9.109 \times 10^{-31}$ kg

proton mass ................ $m_{\mathrm{p}} = 1.673 \times 10^{-27}$ kg

1 bohr............................ $a_0 = 0.5292$ Å

1 hartree ....................... $E_{\mathrm{h}} = 27.21$ eV

Avogrado constant ..... $N_{\mathrm{A}} = 6.022 \times 10^{23}$ mol$^{-1}$

gravitational constant.. $G = 6.674 \times 10^{-11}$ m$^3$ / kg s$^2$

## EQUATIONS THAT MAY BE HELPFUL

### QUANTUM MECHANICS:

Spin operators for spin-$\frac{1}{2}$ particles:

$$\hat{S}_x = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \hat{S}_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \hat{S}_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Normalized stationary states of harmonic oscillator:

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-\xi^2/2} \; ; \quad \psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}} \xi e^{-\xi^2/2} \; , \quad \text{where } \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

### ELECTROSTATICS:

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\mathrm{enc}}}{\varepsilon_0} \qquad \mathbf{E} = -\nabla V$$

$$\int_{r_1}^{r_2} \mathbf{E}\cdot d\boldsymbol{\ell} = V(r_1) - V(r_2) \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

work done $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[V(\mathbf{b}) - V(\mathbf{a})\right]$

Multipole expansion:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos(\theta')\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left\{\tfrac{3}{2}\cos(\theta') - \tfrac{1}{2}\right\}\rho(\mathbf{r}')d\tau' + \ldots\right]$$

where the first term is the monopole term, the second is the dipole term, the third is the quadru-pole term … ; $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ are field point and source point and $\theta'$ is the angle between them.

$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \qquad \nabla\cdot\mathbf{D} = \rho_{\mathrm{f}} \qquad \rho_{\mathrm{b}} = -\nabla\cdot\mathbf{P} \qquad \sigma_{\mathrm{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$

The above are true for *all* dielectrics. Confining ourselves to linear, isotropic, and homogeneous (LIH) dielectrics, we also have:

$\mathbf{D} = \varepsilon\mathbf{E} \qquad \mathbf{P} = \chi_{\mathrm{e}}\varepsilon_0\mathbf{E} \qquad \varepsilon = \varepsilon_0(1+\chi_{\mathrm{e}}) \qquad \kappa_{\mathrm{e}} = \varepsilon/\varepsilon_0 \qquad \chi_{\mathrm{e}} = \kappa_{\mathrm{e}} - 1$

$C(\text{dielectric}) = \kappa_{\mathrm{e}}C(\text{vacuum})$

Boundary conditions: $E_{2\mathrm{t}} - E_{1\mathrm{t}} = 0$ , $E_{2\mathrm{n}} - E_{1\mathrm{n}} = \dfrac{\sigma}{\varepsilon_0}$

**MAGNETOSTATICS:**

Lorentz Force: $\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) + q\mathbf{E}$ Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$ , $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{Id\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$) .

For surface currents: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{K}\times\hat{\mathbf{R}}}{R^2}da$ .

For straight wire segment: $B = \dfrac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$ where $s$ is perpendicular distance from wire.

Infinitely long solenoid: $B$-field inside is $B = \mu_0 nI$ ($n$ is number of turns per unit length).

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enclosed}}$ .

**Magnetic vector potential A**

$\mathbf{B} = \nabla \times \mathbf{A}$

$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \Rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r - r'} \, d\tau'$

For line and surface currents $\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r - r'} d\ell \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r - r'} da'$

From Stokes' theorem $\oint \mathbf{A} \cdot d\ell = \int \mathbf{B} \cdot d\mathbf{a}$

For a magnetic dipole $\mathbf{m}$, $\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$

**Magnetic dipoles**

Magnetic dipole moment of a current distribution is given by $\mathbf{m} = I \int d\mathbf{a}$.

Force on magnetic dipole $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$

Torque on magnetic dipole $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$

*B*-field of magnetic dipole $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \right]$

Dipole-dipole interaction energy is $U_{\text{DD}} = \frac{\mu_0}{4\pi R^3} \left[ (\mathbf{m}_1 \cdot \mathbf{m}_2) - 3(\mathbf{m}_1 \cdot \hat{\mathbf{R}})(\mathbf{m}_2 \cdot \hat{\mathbf{R}}) \right]$, where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$

**Material with magnetization M**

produces a magnetic field equivalent to that of (bound) volume and surface current densities

$\mathbf{J}_{\text{b}} = \nabla \times \mathbf{M}$ and $\mathbf{K}_{\text{b}} = \mathbf{M} \times \hat{\mathbf{n}}$.

$\oint \mathbf{H} \cdot d\ell = I_{\text{free, enclosed}} \qquad \mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M}$

For linear magnetic material $\mathbf{M} = \chi_{\text{m}} \mathbf{H}$ and $\mathbf{B} = \mu_0 (1 + \chi_{\text{m}}) \mathbf{H}$ or $\mathbf{B} = \mu \mathbf{H}$

Boundary conditions: $B_{2\text{n}} - B_{1\text{n}} = 0 \qquad B_{2\parallel} - B_{1\parallel} = \mu_0 K$

**Maxwell's Equations in vacuum:**

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

**Maxwell's Equations in linear, isotropic, and homogeneous (LIH) media:**

1. $\nabla \cdot \mathbf{E} = \dfrac{\rho_{\mathrm{f}}}{\varepsilon}$ Gauss' Law
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ Faraday's Law
4. $\nabla \times \mathbf{B} = \mu \mathbf{J}_{\mathrm{f}} + \varepsilon \mu \dfrac{\partial \mathbf{E}}{\partial t}$ Ampere's Law with Maxwell's correction

Alternative way of writing Faraday's Law: $\oint \mathbf{E} \cdot d\ell = -\dfrac{d\Phi}{dt}$

Mutual and self inductance: $\Phi_2 = M_{21} I_1$, and $M_{21} = M_{12}$; $\Phi = LI$

Energy stored in electric, magnetic field:

$$W = \tfrac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2 / 2C$$

$$W = \tfrac{1}{2}\mu_0^{-1} \int_V B^2 d\tau = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \oint \mathbf{A} \cdot \mathbf{I} \, d\ell$$

## VECTOR DERIVATIVES

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence :* $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

*Curl :* $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

*Laplacian :* $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence :* $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

*Curl :* $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian :* $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

*Gradient :* $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence :* $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

*Curl :* $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

*Laplacian :* $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

## VECTOR IDENTITIES

**Triple Products**

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

**Product Rules**

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

**Second Derivatives**

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem :** $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem :** $\int (\nabla \cdot \mathbf{A})\,d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem :** $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

**INTEGRALS**

| $f(x)$ | $\int_0^\infty f(x)\,dx$ |
|---|---|
| $e^{-ax^2}$ ............... | $\frac{\sqrt{\pi}}{2\sqrt{a}}$ |
| $xe^{-ax^2}$ ............... | $\frac{1}{2a}$ |
| $x^2 e^{-ax^2}$ ............... | $\frac{\sqrt{\pi}}{4a^{3/2}}$ |
| $x^3 e^{-ax^2}$ ............... | $\frac{1}{2a^2}$ |
| $x^4 e^{-ax^2}$ ............... | $\frac{3\sqrt{\pi}}{8a^{5/2}}$ |
| $x^5 e^{-ax^2}$ ............... | $\frac{1}{a^3}$ |
| $x^6 e^{-ax^2}$ ............... | $\frac{15\sqrt{\pi}}{16a^{7/2}}$ |

$$\int_0^\infty \frac{1}{1+ay^2}dy = \pi / 2a^{1/2}$$

$$\int_0^\infty y^n e^{-y/a} dy = n!a^{n+1}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2}$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2)$$

$$\int \frac{dx}{x(a^2+x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2+x^2}\right)$$

$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right)$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right)$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2$$