**Vector algebra**:

$\|\mathbf{u}\| = \sqrt{(\mathbf{x\hat{i}})^2 + (\mathbf{y\hat{j}})^2 + (\mathbf{z\hat{k}})^2}$

$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

Unit vector: $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$

$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

$(\boldsymbol{\alpha}\mathbf{u})\cdot\mathbf{v} = \boldsymbol{\alpha}(\mathbf{u}\cdot\mathbf{v})$

$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = \mathbf{1}$

$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \mathbf{0}$

$(\alpha\mathbf{u} + \beta\mathbf{v})\cdot\mathbf{w} = \alpha(\mathbf{u}\cdot\mathbf{w}) + \beta(\mathbf{v}\cdot\mathbf{w})$

Magnitude of $\mathbf{u}\times\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta\hat{\mathbf{e}}$

$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}, \hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}}$

$(\boldsymbol{\alpha}\mathbf{u})\times\mathbf{v} = \boldsymbol{\alpha}(\mathbf{u}\times\mathbf{v})$

$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$

$\|\mathbf{u}\times\mathbf{v}\|$ = area of $\mathbf{u},\mathbf{v}$ parallelogram

$\mathbf{M}\equiv\mathbf{R}\times\mathbf{F}$, Moment $\mathbf{M}$ of $\mathbf{F}$ about $\mathbf{P}$

$\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \mathbf{0}$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_x & v_z & v_z\end{vmatrix} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} + (u_3v_1 - u_1v_3)\hat{\mathbf{j}} + (u_2v_1 - u_2v_1)\hat{\mathbf{k}}$$

$\mathbf{u}\times\mathbf{v} = u_\mathbf{i}v_\mathbf{j}\varepsilon_{\mathbf{ijk}}e_\mathbf{k}$ $\quad\varepsilon_{\mathbf{ijk}}$ is the permutation tensor

$e_\mathbf{i}\times e_\mathbf{j} = \varepsilon_{\mathbf{ijk}}e_\mathbf{k}$

$\mathbf{R}\cdot\hat{\mathbf{n}} = \|\mathbf{R_0}\|$, $\quad$ R is any point plane, $\|\mathbf{R_0}\|$ is shortest distance to plane

Plane: $ax + by + cz = d$, $(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$ is normal vector (not unit normal)

Shortest distance $\mathrm{D} = \frac{|\mathrm{d}|}{\sqrt{a^2+b^2+c^2}}$

$|\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}|$ = volume of $\mathbf{u},\mathbf{v},\mathbf{w}$ parallelepiped

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \begin{vmatrix}u1 & u2 & u3\\ v1 & v2 & v3\\ w1 & w2 & w3\end{vmatrix}$$

$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$ (if $= 0$, $\mathbf{u},\mathbf{v},\mathbf{w}$ are LD, in plane)

$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$

$(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w})' = \mathbf{u'}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v'}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\mathbf{w'}$

$[(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})]' = (\mathbf{u'}\times(\mathbf{v}\times\mathbf{w}) + (\mathbf{u}\times(\mathbf{v}'\times\mathbf{w}) + (\mathbf{u}\times(\mathbf{v}\times\mathbf{w'})$

$\|\mathbf{u}\| = \frac{\mathbf{u}\cdot\mathbf{u'}}{\|\mathbf{u}\|}$

$\mathbf{u_i}\boldsymbol{\delta}_{\mathbf{ij}} = \mathbf{u_j}$

$\mathbf{u}\cdot\mathbf{v} = \mathbf{u_i e_i}\cdot\mathbf{v_j e_j} = \mathbf{u_i v_j e_i}\cdot\mathbf{e_j} = \mathbf{u_i v_j}\boldsymbol{\delta}_{\mathbf{ij}}$

$\mathbf{u}\cdot\mathbf{v} = \mathbf{u^T v}$

**Cartesion**:

$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\mathbf{v}(t) = x'\hat{\mathbf{i}} + y'\hat{\mathbf{j}} + z'\hat{\mathbf{k}}$

$\mathbf{a}(t) = x''\hat{\mathbf{i}} + y''\hat{\mathbf{j}} + z''\hat{\mathbf{k}}$

$$dA = \begin{cases} dy\,dz & (constant\ x\ surface)\\ dx\,dz & (constant\ y\ surface)\\ dy\,dx & (constant\ z\ surface)\end{cases}$$

Vector differential operator (Gradient): $\boldsymbol{\nabla}\equiv\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$

Div $\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$\mathbf{v}\cdot\boldsymbol{\nabla} = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}$

Grad $u\equiv\boldsymbol{\nabla}u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}$, $\quad$ scalar field $u$ to vector field

$$\text{Curl } \mathbf{v}\equiv\boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}},$$
vector field $\mathbf{v}$ to vector field

**Polar**: $\hat{\mathbf{e}}_\mathbf{r}(\theta)$, $\hat{\mathbf{e}}_{\boldsymbol{\theta}}(\theta)$

$\mathbf{R} = r\hat{\mathbf{e}}_r$

$\mathbf{v}(\mathrm{t}) = r'\hat{\mathbf{e}}_\mathbf{r} + r\theta'\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\mathbf{a}(t) = (r'' - r\theta'^2)\hat{\mathbf{e}}_\mathbf{r} + (r\theta'' + 2r'\theta')\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\frac{\partial\hat{\mathbf{e}}_\mathbf{r}}{\partial\theta} = \hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\frac{\partial\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{e}}_\mathbf{r}$

$\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{e}}_\mathbf{r} - \sin\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\hat{\mathbf{j}} = \sin\theta\hat{\mathbf{e}}_\mathbf{r} + \cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$x = r\cos\theta$, $y = r\sin\theta$

**Cylindrical**: $\hat{\mathbf{e}}_\mathbf{r}(\theta)$, $\hat{\mathbf{e}}_{\boldsymbol{\theta}}(\theta)$, $\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{k}}$

$\mathbf{R} = r\hat{\mathbf{e}}_\mathbf{r} + z\hat{\mathbf{e}}_\mathbf{z}$, $\quad d\mathbf{R} = dr\hat{\mathbf{e}}_\mathbf{r} + rd\theta\hat{\mathbf{e}}_{\boldsymbol{\theta}} + dz\hat{\mathbf{e}}_\mathbf{z}$

$\mathbf{v}(t) = r'\hat{\mathbf{e}}_\mathbf{r} + r\theta'\hat{\mathbf{e}}_{\boldsymbol{\theta}} + z'\hat{\mathbf{e}}_\mathbf{z}$

$\mathbf{a}(t) = (r'' - r\theta'^2)\hat{\mathbf{e}}_\mathbf{r} + (r\theta'' + 2r'\theta')\hat{\mathbf{e}}_{\boldsymbol{\theta}} + z''\hat{\mathbf{e}}_\mathbf{z}$

$\frac{\partial\hat{\mathbf{e}}_\mathbf{r}}{\partial\theta} = \hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\frac{\partial\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{e}}_\mathbf{r}$

$\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{e}}_\mathbf{r} - \sin\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\hat{\mathbf{j}} = \sin\theta\hat{\mathbf{e}}_\mathbf{r} + \cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\hat{\mathbf{e}}_\mathbf{r}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\hat{\mathbf{e}}_{\boldsymbol{\theta}}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{r}$, $\quad\hat{\mathbf{e}}_\mathbf{r}\times\hat{\mathbf{e}}_{\boldsymbol{\theta}} = \hat{\mathbf{e}}_\mathbf{z}$

If taking the cross product, set it up as $\hat{\mathbf{e}}_\mathbf{r}\to\hat{\mathbf{e}}_{\boldsymbol{\theta}}\to\hat{\mathbf{e}}_\mathbf{z}$, L to R

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$dA = \begin{cases} r\,d\theta dz & (constant\ r\ surface)\\ dr\,dz & (constant\ \theta\ surface)\\ r\,dr\,d\theta & (constant\ z\ surface)\end{cases}$$

E.g. for a cone $r, \theta$, and $z$ are **not** constant. For a cylinder, $r$ is constant.

$dV = r\,dr\,d\theta\,dz$

$\boldsymbol{\nabla}u = \frac{\partial u}{\partial r}\hat{\mathbf{e}}_\mathbf{r} + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\mathbf{e}}_{\boldsymbol{\theta}} + \frac{\partial u}{\partial z}\hat{\mathbf{e}}_\mathbf{z}$

$\boldsymbol{\nabla}^2u = \frac{\partial^2u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2} + \frac{\partial^2u}{\partial z^2}$

$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \frac{\partial}{\partial z}v_z$

$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{e}}_\mathbf{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\mathbf{e}}_{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right)\hat{\mathbf{e}}_\mathbf{z}$

**Spherical**: $\hat{\mathbf{e}}_{\boldsymbol{\rho}}(\phi,\theta)$, $\quad\hat{\mathbf{e}}_{\boldsymbol{\phi}}(\phi,\theta)$, $\quad\hat{\mathbf{e}}_{\boldsymbol{\theta}}(\theta)$

$\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\rho}}}{\partial\rho} = 0$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\rho}}}{\partial\phi} = \hat{\mathbf{e}}_{\boldsymbol{\phi}}$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\rho}}}{\partial\theta} = \sin\phi\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\phi}}}{\partial\rho} = 0$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\phi}}}{\partial\phi} = -\hat{\mathbf{e}}_{\boldsymbol{\rho}}$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\phi}}}{\partial\theta} = \cos\phi\,\hat{\mathbf{e}}_\theta$

$\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\theta}}}{\partial\rho} = 0$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\theta}}}{\partial\phi} = 0$, $\quad\frac{\partial\hat{\mathbf{e}}_{\boldsymbol{\theta}}}{\partial\theta} = -\sin\phi\,\hat{\mathbf{e}}_{\boldsymbol{\rho}} - \cos\phi\,\hat{\mathbf{e}}_\phi$

$\mathbf{R} = \rho\hat{e}_\rho$, $\quad d\mathbf{R} = d\rho\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \rho d\phi\hat{\mathbf{e}}_{\boldsymbol{\phi}} + \rho\sin\phi\,d\theta\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\mathbf{v}(t) = \rho'\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \rho\phi'\hat{\mathbf{e}}_{\boldsymbol{\phi}} + \rho\theta'\sin\phi\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$$\mathbf{a}(t) = (\rho'' - \rho\phi'^2 - \rho\theta'^2\sin^2\phi)\hat{\mathbf{e}}_{\boldsymbol{\rho}} + (\rho\phi'' + 2\rho'\phi' - \rho\theta'^2\sin\phi\cos\phi)\hat{\mathbf{e}}_{\boldsymbol{\phi}} + (\rho\theta''\sin\phi + 2\rho'\phi'\sin\phi + 2\rho\theta'\phi'\cos\phi)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$$

$$dA = \begin{cases} \rho^2|\sin\phi|d\phi\,d\theta & (constant\ \rho\ surface)\\ \rho|\sin\phi|d\rho\,d\theta & (constant\ \phi\ surface)\\ \rho\,d\rho\,d\phi & (constant\ \theta\ surface)\end{cases}$$

$dV = \rho^2|\sin\phi|d\rho\,d\phi\,d\theta$

$\hat{\mathbf{e}}_{\boldsymbol{\theta}}\times\hat{\mathbf{e}}_\mathbf{r} = \hat{\mathbf{e}}_{\boldsymbol{\phi}}$, $\quad\hat{\mathbf{e}}_{\boldsymbol{\phi}}\times\hat{\mathbf{e}}_\mathbf{r} = -\hat{\mathbf{e}}_{\boldsymbol{\theta}}$, $\quad\hat{\mathbf{e}}_{\boldsymbol{\phi}}\times\hat{\mathbf{e}}_{\boldsymbol{\theta}} = \hat{\mathbf{e}}_\mathbf{r}$

$\hat{\mathbf{e}}_{\boldsymbol{\rho}} = \sin\phi\,(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + \cos\phi\,\hat{\mathbf{k}}$

$\hat{\mathbf{e}}_{\boldsymbol{\phi}} = \cos\phi\,(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) - \sin\phi\,\hat{\mathbf{k}}$

$\hat{\mathbf{e}}_{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$

$\hat{\mathbf{i}} = \sin\phi\cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \cos\phi\cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} - \sin\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\hat{\mathbf{j}} = \sin\phi\sin\theta\,\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \cos\phi\sin\theta\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} + \cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\hat{\mathbf{k}} = \cos\phi\,\hat{\mathbf{e}}_{\boldsymbol{\rho}} - \sin\phi\,\hat{\mathbf{e}}_{\boldsymbol{\phi}}$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\boldsymbol{\nabla}u = \frac{\partial u}{\partial\rho}\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial u}{\partial\phi}\hat{\mathbf{e}}_{\boldsymbol{\phi}} + \frac{1}{\rho\sin\phi}\frac{\partial u}{\partial\theta}\hat{\mathbf{e}}_{\boldsymbol{\theta}}$

$\boldsymbol{\nabla}^2u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2u}{\partial\theta^2}\right]$

$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2v_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(v_\phi\sin\phi) + \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial\theta}$

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{1}{\rho\sin\phi}\left(\frac{\partial}{\partial\phi}(v_\theta\sin\phi) - \frac{\partial v_\phi}{\partial\theta}\right)\hat{\mathbf{e}}_{\boldsymbol{\rho}} + \frac{1}{\rho}\left(\frac{1}{\sin\phi}\frac{\partial v_\rho}{\partial\theta} - \frac{\partial(\rho v_\theta)}{\partial\rho}\right)\hat{\mathbf{e}}_{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho v_\phi)}{\partial\rho} - \frac{\partial v_\rho}{\partial\phi}\right)\hat{\mathbf{e}}_{\boldsymbol{\theta}}$$

**Curves and line integrals**

Arc legth $s(\tau) = \int_{\tau_0}^{\tau}\sqrt{\mathbf{R}'(t)\cdot\mathbf{R}'(t)}dt$

Line Integral: $\int_C f(x,y,z)ds = \int_a^b f(x(\tau),y(\tau),z(\tau))\sqrt{\mathbf{R}'(\tau)\cdot\mathbf{R}'(\tau)}d\tau$

**Note:** **R** is the vector that traces out the curve. For example, if the curve is a semicircle in the quadrant I and II, then $\mathbf{R}(\boldsymbol{\theta}) = r\hat{\mathbf{e}}_{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$, where $0 \le \theta \le \boldsymbol{\pi}$. And in this case, $\tau = \theta$.

**Parameterization of a straight line:**

$$x = x_1 + (x_2 - x_1)\tau, \qquad y = y_1 + (y_2 - y_1)\tau, \qquad z = z_1 + (z_2 - z_1)\tau, \qquad (0 \le \tau < 1)$$

**Parameterization:** The goal is to solve for your curve or surface in terms of a variable that suits your preferred coordinate system. If your curve is an intersection of two surfaces, then solve for the intersection just like you would a system of equations (Gauss elimination, etc.), but before you solve, chose a parameterization that makes sense for the surfaces in question. I.e. if you have a plane intersecting a cylinder, chose θ as the parameter (from 0 to 2π), make the appropriate substitutions $x = r\cos\theta$, $y = r\sin\theta$ for the cylinder and then solve for x,y, and z in terms of the new parameter.

**Net work done traversing curve C:** $\int_C \mathbf{v}\cdot d\mathbf{R} = \int_C f dS$,

Where **v** is a force vector field and **R** is the position vector to some reference point.

$$\int_C \mathbf{v}\cdot d\mathbf{R} = \int_C (v_x(x,y,z)dx + v_y(x,y,z)dy + v_z(x,y,z)dz),$$

Note 1: The dot turns this from two vectors to a scalar.

Note 2: If the curve is not continuous, need to break up the integral.

**Stoke's Theorem:** $\oint_C \mathbf{v}\cdot d\mathbf{R} = \oint_S \hat{\mathbf{n}}\cdot\nabla\times\mathbf{v}dA,$

Note 1: If the surface is not closed, the $\hat{\mathbf{n}}$ follows the right hand rule.

Note 2: The line integral must be closed.

**Greene's Theorem:** $\int_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_C Pdx + Qdy$

Note 1: Vector field: $\mathbf{v} = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$

Note 2: Edge of S must be piecewise, smooth, simple, closed, oriented CC.

**Greene's 1st identity:** $\int_V (\nabla u\cdot\nabla v + u\nabla^2 v)dV = \int_S u\frac{\partial v}{\partial n}dA$

**Greene's 2nd identity:** $\int_V (u\nabla^2 v\cdot\nabla v - v\nabla^2 u)dV = \int_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dA$

**Note 1:** $u\frac{\partial v}{\partial n} = u\hat{\mathbf{n}}\cdot\nabla v$

**Note 2:** $u$ and $v$ are scalar fields

**Divergence Theorem:** $\int_v \nabla\cdot\mathbf{v}\, dV = \int_S \hat{\mathbf{n}}\cdot\mathbf{v}\, dA$

$$\int_S \hat{\mathbf{n}}udA = \int_V \nabla udV, \quad \int_S \hat{\mathbf{n}}\times\mathbf{v}d\boldsymbol{A} = \int_V \nabla\times\mathbf{v}dV$$

Where: **v** is a vector field, and $\hat{\mathbf{n}}$ is the unit normal vector to the surface.

If you have several discontinuous surfaces forming one piecewise smooth surface, you need to integrate each one seperately, and then add them.

The unit normal vectors always point outward from the surface.

$\hat{\mathbf{n}} = \pm\frac{\nabla g}{\|\nabla g\|}$, Where $g = g(x,y,z)$, or $g(\rho,\phi,\theta)$, $etc.$ = a surface

Example: We have the equation for a paraboloid: $x^2 + y^2 = z$. First, get all the variables to one side, so: $x^2 + y^2 - z = 0$. This is now $g(x,y,z)$. The gradient of $g$ is now the normal to the surface. This is a "level surface". If we want to find the normal to a "level curve", then we set $x$,$y$,or $z$ to a constant and the take the gradient, e.g.: $1^2 + y^2 - z = 0$. This is now our $g(x,y,z)$.

Distance between **x** and **x'**: $d(\mathbf{x},\mathbf{x}') = \sqrt{(x_1 - x_1')^2 + \cdots}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Chain rule: $\frac{dF}{dt} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}; F = f(x(t),y(t))$

Leibniz rule: $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx$

$$= \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}f(x,t)dx + b'(t)f(b(t),t) - a'(t)f(a(t),t)$$

Jacobian Matrix: $\frac{\partial(y_1,\ldots,y_m)}{\partial(x_1,\ldots,x_n)} = \begin{matrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{matrix}$

Div $\mathbf{v} \equiv \nabla\cdot\mathbf{v}$, vector field **v** to scalar field

Physical significance of curl: if **v** is a fluid velocity field, then $\nabla\times\mathbf{v}$ at any point P is twice the angular velocity of the fluid at P.

If Curl $\mathbf{v} = 0$, we have irrotation field

Div curl $\mathbf{v} = \nabla\cdot\nabla\times\mathbf{v} = 0$

curl grad $u = \nabla\times\nabla u = 0$

$\nabla\cdot\nabla = \nabla^2$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\nabla\mathbf{x} = \hat{\mathbf{i}}, \nabla\mathbf{y} = \hat{\mathbf{j}}$

$\nabla\cdot(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\nabla\cdot\mathbf{u} + \beta\nabla\cdot\mathbf{v}$

$\nabla(\alpha u + \beta v) = \alpha\nabla u + \beta\nabla v$

$\nabla\times(\alpha\mathbf{u} + \beta\boldsymbol{v}) = \alpha\nabla\times\mathbf{u} + \beta\nabla\times\mathbf{v}$

$\nabla\cdot(u\mathbf{v}) = \nabla u\cdot\mathbf{v} + u\nabla\cdot\mathbf{v}$

$\nabla\times(u\mathbf{v}) = \nabla u\times\mathbf{v} + u\nabla\times\mathbf{v}$

$\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\nabla\cdot\nabla v$

$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\times\mathbf{v}$

$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\nabla\cdot\mathbf{v} - \mathbf{v}\nabla\cdot\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v}$

$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times\nabla\times\mathbf{u})$

$u\hat{\mathbf{n}}\cdot\nabla v = u\frac{\partial v}{\partial n}$

$$(\mathbf{u}\cdot\nabla)\mathbf{v} = (\mathbf{u}\cdot\nabla)(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) = \left(u_x\frac{\partial v_x}{\partial} + u_y\frac{\partial v_y}{\partial} + u_z\frac{\partial v_z}{\partial}\right)\hat{\boldsymbol{\imath}} + (etc.)\hat{\boldsymbol{\jmath}} + (etc.)\hat{\boldsymbol{k}}$$

Laplace equation: $\nabla^2\boldsymbol{\varphi} = \nabla\cdot\nabla\boldsymbol{\varphi} = 0$

Poisson equation: $\nabla^2\boldsymbol{\varphi} = F$

Diffusion equation: $\nabla^2\boldsymbol{\varphi} = \frac{\partial\boldsymbol{\varphi}}{\partial t}$

Wave equation: $c^2\nabla^2\boldsymbol{\varphi} = \frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}$

Level Curve: The function parameters that yield a specified "z".

$$\text{div}\,\mathbf{v}(P) \equiv \lim_{B\to 0}\left(\frac{\int_S \hat{\mathbf{n}}\cdot\mathbf{v}dA}{V}\right)$$

**Trig Identities:**

$1 = \cos^2\theta + \sin^2\theta$

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} = \frac{c}{\sin\theta} = 2R$

where $R$ is the radius of the triangle's circumference.

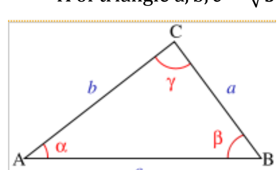
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ 
$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

We can derive all the others from these two and $\sin^2\theta + \cos^2\theta = 1$

Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Length of any one side of a triangle cannot exceed the sum of the lengths of the other two sides.

A of triangle a, b, c $= \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}$, where $s = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{\mathbf{2}}$




**Change of Variables Example:**

**Geometry:**

Plane General Equation: $ax + by + cz = d$, where $a, b, c$ makes a unit normal vector.

Distance from origin: $D = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$

Given 3 points: $(x1,y1,z1), (x2,y2,z2), (x3,y3,z3)$

$$a = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, b = \begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix}\, c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\, d = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

**Sphere**: Equation for a sphere, centered at $x_0, y_0, z_0$, radius $r$: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

Surface area of sphere: $A = 4\pi r^2$

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

**Cylinder**: $ax^2 + by^2 = r$, where $r$ is the radius.
If $a$ and $b$ are 1, then it is a circular cylinder.

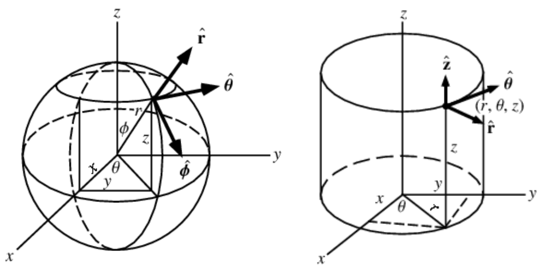
**Ellipse**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is semiminor axis, b is semimajor axis

**Paraboloid**: $z = x^2 + y^2$ (elliptic), $z = x^2 - y^2$ (hyperbolic)

**Parabola (example)**: $z = y^2 + 1$, note that y = r ir we revolve the parabola about z.

**Another parabola**: $x^2 - y^2 = 1$

**Circle**: $x^2 + y^2 = r^2$




**Parameterization of curves, surfaces, and volumes:**

$\mathbf{R}(\tau) = x(\tau)\hat{\mathbf{i}} + y(\tau)\hat{\mathbf{j}} + z(\tau)\hat{\mathbf{k}}$

$\mathbf{R}(u,v) = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}}$

$\mathbf{R}(u,v,w) = x(u,v,w)\hat{\mathbf{i}} + y(u,v,w)\hat{\mathbf{j}} + z(u,v,w)\hat{\mathbf{k}}$

$ds = \sqrt{\mathbf{R}'(\tau)\cdot\mathbf{R}'(\tau)}d\tau$

$dA = \|\mathbf{R}_u \times \mathbf{R}_v\| du dv$

$dV = |\mathbf{R}_u \cdot \mathbf{R}_v \times \mathbf{R}_w| du dv dw$

$\hat{\mathbf{n}} = \frac{\mathbf{R}_u \times \mathbf{R}_v}{\|\mathbf{R}_u \times \mathbf{R}_v\|}$, $\mathbf{R}(u,v)$ is parameterized surface, $\mathbf{R}_u = \frac{\partial \mathbf{R}}{\partial u}$

Computationally, we can express these in terms of the components *x,y,z* of **R:**

$ds = \sqrt{x'^2 + y'^2 + z'^2}d\tau$

$dA = \sqrt{EG - F^2}du\,dv$, where: $E = x_u^2 + y_u^2 + z_u^2$, $G = x_v^2 + y_v^2 + z_v^2$, $F = x_u x_v + y_u y_v + z_u z_v$

Notes: Find x,y, and z in terms of the two parameters u and v. z will be in terms of u and v, e.g. $f(x,y)$. Then take the appropriate derivatives. The limits of integration are over the new parameters u and v.

$dV = \frac{\partial(x,y,z)}{\partial(u,v,w)} du dv dw$, note the Jacobian.

Special cases of the above:

Case 1: surface is flat and in the xy plane: $dA = \frac{\partial(x,y)}{\partial(u,v)} du dv$

Case 2: surface is known of the form $z = f(x,y)$: $dA = \sqrt{1 + f_x^2 + f_y^2} dx dy$

When we integrate Case 2, the limits of integration are defined by the region in the xy plane under the surface.

**Tangent plane** at $x_p, y_p, z_p$ on a surface: $f_x(x_p,y_p,z_p)(x - x_p) + f_y(x_p,y_p,z_p)(y - y_p) + f_z(x_p,y_p,z_p)(z - z_p) = 0$

Note: $f(x,y,z)$ is the function, e.g. if our equation is $x^2 + y^2 = z$, then $f(x,y,z) = x^2 + y^2 - z$. And $f_x$ is $\frac{\partial f}{\partial x}$.

$\hat{\mathbf{n}} = \frac{(f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}})}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$, where $f_x, f_y, f_z$ are evaluated at the point $(x_p, y_p, z_p)$ on $S$.

$$A = \iint_{\Re} \|\mathbf{R}_u \times \mathbf{R}_v\| du dv, \quad dA = \|\mathbf{R}_u \times \mathbf{R}_v\| du dv$$

$$A = \iint_{\Re} \sqrt{1 + f_x^2 + f_y^2} dx dy$$, where $A$ is the area of the surface and $\Re$ is the region in the x, y plane under $S$. We only use this equation if we can write the surface as *z=f(x,y).*

$$A = \iint_{\Re} f(x,y) dA = \int_{y1}^{y2}\int_{x1(y)}^{x2(y)} f(x,y) dx dy$$

$$V = \iiint_{\Re} f(x,y,z) dV = \int_{z1}^{z2}\int_{y1(z)}^{y2(z)}\int_{x1(y,z)}^{x2(y,z)} f(x,y,z) dx dy dz$$

**Matrices**

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$

$\mathbf{AB} = \mathbf{C} = \{c_{ij}\} = \left\{\sum_{k=1}^{n} a_{ik}b_{kj}\right\}$; $(1 \le i \le m, 1 \le j \le p)$

Note sizes of **ABC**: $m \times n$ times $n \times p = m \times p$

$\mathbf{AB} \neq \mathbf{BA}$

$\mathbf{AA}\ldots\mathbf{A} \equiv \mathbf{A}^p$, $\mathbf{A}^p\mathbf{A}^q = \mathbf{A}^{p+q}$, $(\mathbf{A}^p)^q = \mathbf{A}^{pq}$

If $\mathbf{AB} = \mathbf{AC}$, does **not** imply that $\mathbf{B} = \mathbf{C}$

$(\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2$: $(\mathbf{AB})^2 = \mathbf{ABAB}$, $\mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}$

Transpose: switch rows for columns: $(\mathbf{A}^T)^T = \mathbf{A}$

$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$

$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, $(\mathbf{ABCD})^T = \mathbf{D}^T\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$

x and y are column vectors:

$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$

If $\mathbf{A}^T = \mathbf{A}$, then **A** is symmetric.

If $\mathbf{A}^T = -\mathbf{A}$, then **A** is antisymmetric.

Decompose: $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$
$= \mathbf{A_1} + \mathbf{A_2}$, where **A** is square, $\mathbf{A_1}$ is symmetric, and $\mathbf{A_2}$ is antisymmetric.

$\mathbf{A}^2\mathbf{I}$ does not imply that $\mathbf{A} = \pm\mathbf{I}$

$\mathbf{Ax} = \mathbf{c}$, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

$\mathbf{AB} = \mathbf{A}[\mathbf{C_1}\ldots\mathbf{C_n}] = [\mathbf{AC_1}\ldots\mathbf{AC_n}]$ If **B** is paritioned into n columns.

$\det\mathbf{A} = \sum_{k=1}^{n} a_{jk}\mathbf{A}_{jk}$, where $\mathbf{A}_{jk} = (-1)^{j+k}\mathbf{M}_{jk}$

Notes: Fix j, Find the M's (little determinants), carry out the summation.

**Properties of determinants:**

1) If a row or column is modified by adding $\alpha$ times another row, then det**A** does not change.
2) If any two rows are changed then det**A**=-det**B**.
3) If **A** is triangular, then det**A** is the product of the diagonal.
4) If a row or column is 0 then the det is 0.
5) If a row or a column is a linear combo of other rows or columns then the det = 0
6) $\det(\alpha\mathbf{A}) = \alpha\det\mathbf{A}$
7) $\det(\mathbf{A}^T) = \det\mathbf{A}$
8) $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$
9) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
10) If any two rows are equal, det=0.
11) If det $\neq 0$, then **A**x=**c** has a unique solution.
12) If det=0 then **A** is singular.
13) If m $< n$ then system is underdetermined.
14) If m $> n$ then system is overdetermined.
15) Underdetermined and 2 unknowns→ 2 parameter family of solutions and solutions lie in a plane. 1 parameter family and solutions like on a line, etc.
16) Inconsistent system: 0= -15 for a solution (for example).
17) Scaling a row or column scales the determinant.

$\mathbf{A} = \begin{bmatrix} \boxed{\mathbf{A_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boxed{\mathbf{A_m}} \end{bmatrix}$, $\det\mathbf{A} = (\det\mathbf{A_1})(\ldots)(\det\mathbf{A_m})$

$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}\right)$

$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\operatorname{adj}\mathbf{A} = \frac{1}{\det\mathbf{A}}\begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix}$,

adj**A** is the **transpose** of the cofactor matrix.

Recall that the cofactor is: $\mathbf{A}_{jk} = (-1)^{j+k}\mathbf{M}_{jk}$. Remember to take the **Transpose** of $\mathbf{A}_{jk}$ to get the adj**A**

Another way to find $\mathbf{A}^{-1}$: Augment **A|I**, then use elementary ops to get $\mathbf{I}|\mathbf{A}^{-1}$

**Inverses**: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$, $\det(\mathbf{A}^{-1}) = \frac{\mathbf{1}}{\det\mathbf{A}}$

$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{p-1}$,
where p is the power that yields $\mathbf{A}^p = 0$ (Nilpotent).

If **A** is invertible, then **AB**=**AC** implies that **B**=**C**, **BA**=**CA** implies that **B**=**C**, **AB**=0 implies that **B**=0.

$\boldsymbol{A}^{-1} = \begin{bmatrix} \boxed{\mathbf{A^{-1}}_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boxed{\mathbf{A^{-1}}_m} \end{bmatrix}$

Theorem:
If A is $n \times n$ and $\det\mathbf{A} \neq 0$, then $\mathbf{Ax} = \mathbf{c}$ admits the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$.

$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}$

**Cramer's Rule**: If **A**x=**c** where A is invertible, then each component $x_i$ of **x** may be computed as the ratio of two determinants; the denominator is det **A**, and the numerator is det **A** but with the *i*th column replaced by **c**.

$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Notes: Decompose **A** into **LU** (e.g. start with $u_{11} = a_{11}$, *then* $u_{11} l_{21} = a_{21}$, etc.), then solve **Ly=c** for **y**, then solve **Ux**=**y** for **x**.

**Ill conditioned**: Small changes in matrix elements yield large changes in det, inverse, etc. To test if $\frac{\det \mathbf{A}}{\sum_j^n \sum_i^n a_{ji}} \ll 1$ then the matrix is ill conditioned.

**Vector Spaces:**

N − space: $\mathbb{R}^n = (a_1, a_2, \dots, a_n)$ This means we have a vector with "*n*" dimesions.

Subspace: If a subset T of a vector space is itself a vector space (with the same definitions as S for vector addition u+v, scalar multiplication au, zero vector 0, and negative vector –u), then T is a subspace of S.

**Dot product, norm, and angle for n-space**

$$\mathbf{u} \cdot \mathbf{v} \equiv u_1 v_1 + \dots + u_n v_n = \sum_{j=1}^{n} u_j v_j$$

We can "weight" each component:

$$\mathbf{u} \cdot \mathbf{v} \equiv w_1 u_1 v_1 + \dots + w_n u_n v_n = \sum_{j=1}^{n} w_j u_j v_j$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)$$

$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\|$

Orthonormal: a set of vectors where each vector is normalized and each vector in one set is orthogonal to every other vector in the other set. This can be described as the Kronecker delta $= \mathbf{u_i} \cdot \mathbf{v_j} = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$

Norms:

Taxicab norm: $\|\mathbf{u}\| = \sum_{j=1}^{n} |u_j|$

$\|\mathbf{u}\| \equiv \sqrt{\sum_{j=1}^{n} w_j u_j^2}$, if we are using weights.

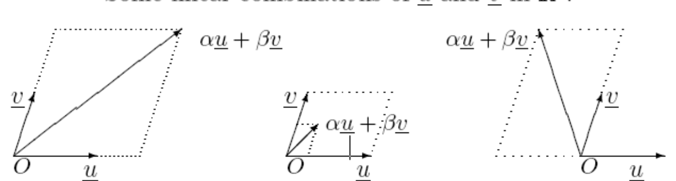
$$\|\mathbf{u}\| \equiv \sqrt{\sum_{j=1}^{n} u_j^2}$$

If vector space consists of functions, say, $u(x), v(x)$ then the inner product is:

$\mathbf{u} \cdot \mathbf{v} = \langle u(x), v(x) \rangle \equiv \int_{\mathbf{a}}^{\mathbf{b}} u(x) v(x) dx$, where a and b are the bounds of the function. Note that the norm can be found for a vector $\|\mathbf{u}\| = \|u(x)\| = \sqrt{\int_0^1 u^2(x) dx}$

**Span**: The set of all linear combinations of the vectors in a vector space is called the span. E.g. for a vector space $\mathrm{u}_1, \dots, \mathrm{u}_\mathrm{k}$, the span is $\alpha_1 \mathrm{u}_1, \dots, \alpha_\mathrm{k} \mathrm{u}_\mathrm{k}$ and is denoted as span$\{\mathrm{u}_1, \dots, \mathrm{u}_\mathrm{k}\}$. The span is a subspace of S. The span has to include the origin.

Some linear combinations of $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^3$:



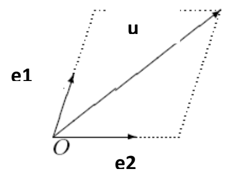

$(\alpha, \beta > 0)$ $(0 < \alpha, \beta < 1)$ $(\alpha < 0 < \beta)$

The above example shows a few linear combinations of the vectors *u* and *v*; the span of this vector space would include **all** the linear combinations (a plane).

- To discover if two spans are equal, say span$_1$ $\{\mathbf{u_1}, \mathbf{u_2}\}$and span$_2\{\mathbf{v_1}, \mathbf{v_2}\}$, write the equation $(\alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}) = \mathbf{w}$, and $(\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2}) = \mathbf{w}$ and solve for $\alpha_1$and $\alpha_2$ in terms of **w.** Compare the **w** vectors. E.g. (for a $\Re^3$vectors):

$\begin{bmatrix} u_{1_1} & u_{2_1} \\ u_{1_2} & u_{2_2} \\ u_{1_3} & u_{2_3} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. In this case we have a non-square matrix, so reduce using elementary operations so that the last row of **A** is zero (keeping track of the operations on **w**). This gives us an equation only in terms of **w**, e.g. $0 = cw_1 + dw_2 + ew_3$

**Bases:**




- $\mathbf{u} = \alpha_1 \mathbf{e_1} + \dots + \alpha_k \mathbf{e_k}$ . A set of **e**'s is a basis for **u** iff it is LI and span S.
- Orthogonal basis are preferred. Given orthog. basis vectors: $\{\mathbf{e_1}, \dots \mathbf{e_k}\}$, suppose we wish to expand a given **u** in terms of these, then

$\mathbf{u} = \alpha_1 \mathbf{e_1} + \dots + \alpha_k \mathbf{e_k}$, where $\alpha_1 = \left(\frac{\mathbf{u} \cdot \mathbf{e_1}}{\mathbf{e_1} \cdot \mathbf{e_1}}\right) \mathbf{e_1}$,

**Orthogonalization process:** Given *k* LI vectors $\mathbf{v_1}, \dots, \mathbf{v_k}$ we can get *k* ON vectors $\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_\mathbf{k}$ in span$\{\mathbf{v_1}, \dots \mathbf{v_k}\}$by:

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}, \quad \hat{\mathbf{e}}_2 = \frac{\mathbf{v_2} - (\mathbf{v_2} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1}{\|\mathbf{v_2} - (\mathbf{v_2} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1\|}, \quad \hat{\mathbf{e}}_j = \frac{\mathbf{v_j} - \sum_{i=1}^{j-1}(\mathbf{v_j} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i}{\left\|\mathbf{v_j} - \sum_{i=1}^{j-1}(\mathbf{v_j} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i\right\|}$$

**Dimensions:** The dimension of a vector space is the greatest number of LI vectors in that vector space. If a vector space contains only a zero vector, the dimension is 0. Dimension relates to bases: The number of basis vectors equals the dimension (because the bases are LI). The dim of a space will be no greater than *n* ($\Re^n$).

**Linear Independence:** A set of vectors is LD if at least one of them can be expressed as a linear combination of the ***others***. Example: $(1,0), (1,1),$ and $(5,4)$are LD ($\mathbf{u_3} = \mathbf{u_1} + \mathbf{4u_2}$).

A set of vectors is LD iff there exist scalars, not all zero, such that $\alpha_1 \mathbf{u_1} + \dots + \alpha_k \mathbf{u_k} = 0$. To solve for the scalars: 1) Set up a system of equations: $\mathbf{A\alpha = 0}$. Where **A** is the matrix of the vectors. 2) Use elementary row operations to reduce to "row echelon form" or as close to it. The non-zero rows are LI.

- A set containing the zero vector is LD.
- Every orthogonal set of nonzero vectors is LI.

**Best Approximations:**

If **u** is any vector with $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

$$\mathbf{u} \approx \sum_{j=1}^{\mathbf{N}} (\mathbf{u} \cdot \hat{\mathbf{e}}_\mathbf{j})\hat{\mathbf{e}}_\mathbf{j}$$

Where we are given **u** and an orthonormal basis set $\{\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_\mathbf{N}\}$. The more basis vectors we use, the closer our approximation will be (up to the number of bases equal to the dimension of **u.)** The error of our approximation is:

$$\|\mathbf{E}\|^2 = \|\mathbf{u}\|^2 - \sum_{j=1}^{N} \alpha_j^2$$

Where: $\alpha_j = \mathbf{u} \cdot \hat{\mathbf{e}}_\mathbf{j}$

**Row-echelon form:**

1) In each row not made up entirely of zeros, the first nonzero element is a 1.
2) In any two consecutive rows not made up entirely of zeros, the leading 1 in the lower row is to the right of the leading 1 in the upper row.
3) If a column contains a leading 1, every other element in that column is a zero.
4) All rows made up entirely of zeros are grouped together at the bottom of the matrix.

**Rank:**

**1)** A matrix (maybe not square) is of rank *r* if it contains at least one $r \times r$ submatrix with nonzero determinant but no square submatrix larger than $r \times r$ with nonzero determinant. You can swap rows and columns to find these submatrices. The zero matrix is of rank 0.
**2)** Elementary row operations do not alter the rank.
**3)** # of LI row vectors = # of LI column vectors = rank.

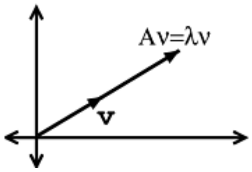
**Elementary operations:**

1) Operate on the augmented matrix (glue **c** onto **A**).
2) Addition of a multiple of one row to another.
3) Multiplication of a row by a nonzero constant.
4) Interchange of two rows.

**Terminology**

1) Consistent: one or more solutions
2) Unique: only one solution
3) Non-unique: more than one solution
4) Inconsistent: No solutions.
5) If m<n: Consistent or inconsistent. If consistent no unique solution exists (p parameter family, where $n - m \le p \le n$.
6) If m>n: Consistent or inconsistent. Can have unique or non-unique solution. (p parameter family, where $1 \le p \le n$).

**Eigenvalue Problem**




If $v$ is an eigenvector of A, then $v$ lies on the vector A$v$. In other words, if $v$ is an eigenvector of A, then A$v$ is the same as some constant times A, e.g. $\lambda v$.

$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$, Characteristic equation: $\det(\mathbf{A}-\lambda\mathbf{I})=0$
To find eigenvalues, solve the characteristic equation for $\lambda$. To find the eigenspaces, 1) find the eigenvalues $\lambda$, 2) for each $\lambda$, plug back into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0},$ and solve for **x.** This will give us an eigenvector for each eigenvalue. 3) We should end up with at least one arbitrary solution (0=0) for each eigenvector. This will give us our arbitrary constants $(\alpha,\beta,\gamma,etc.)$ that when multiplied by the eigenvector gives us our eigenspace.

**Symmetric Matrices:**

- If **A** is **symmetric**, then all of its eigenvalues are real.
- If an $\lambda$ of a **symmetric** matrix **A** is of multiplicity *k*, then the eigenspace corresponding to $\lambda$ is of dimension *k*.
- If **A** is **symmetric,** then eigenvectors corresponding to distinct eigenvalues are orthogonal.
- If an $n\times n$ matrix **A** is symmetric, then its eigenvectors provide an orthogonal basis for *n*-space.
- If **A** is symmetric, then

$$\lambda=\frac{\mathbf{e}^{\mathrm{T}}\mathbf{A}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\mathbf{e}}$$

Where **e** is an eigenvector of **A.**
**Diagonalization:** $\mathbf{Q^{-1}AQ=D}$
Especially useful when solving a system of differential equations. Given a system $\mathbf{Ax=x'}$, our goal is to solve for **x** by 1) Find the **Q** and **D** matrices. **Q** has the eigenvectors for rows, **D** has the eigenvalues in the diagonal (the order of the eigenvalues matches the order of the eigenvectors in **Q**). 2) Write $\tilde{\mathbf{x}}'=\mathbf{D}\tilde{\mathbf{x}}.$ 3) This gives you uncoupled equations, so you can do things like: $\tilde{x}'=\lambda\tilde{x}\rightarrow\tilde{x}=Ce^{\lambda t}$. 4) Now that you have $\tilde{\mathbf{x}}$, you can solve for **x,** by $\mathbf{x}=\mathbf{Q}\tilde{\mathbf{x}}.$

- Every symmetric matrix is diagonalizable.
- If an $n\times n$ matrix has *n* distinct eigenvalues, then it is diagonalizable. (It may be diagonalizable anyway—does not read iff).
- If an $n\times n$ matrix has eigenvalues, then the corresponding eigenvectors are LI.
- **A** is diagonalizable iff it has *n* LI eigenvectors.
- $\mathbf{A^m=QD^mQ^{-1}}$

**Quadratic Form/Cononical Form:** A quadratic form is said to be canonical if all mixed terms (such as $x_1x_2$) are absent.
Example: reduce $f(x_1,x_2)=a_{11}x_1^2+a_{22}x_2^2+(a_{12}+a_{21})x_1x_2$ to canonical form.
1) Identify the **A** matrix: $\mathbf{A}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$, chose $a_{12},a_{21}$ to be equal so that the matrix is symmetrical. 2) Find the eigenvalues. 3) Plug in to get the canonical form: $f(\tilde{x}_1,\tilde{x}_2)=\lambda_1\tilde{x}_1^2+\lambda_2\tilde{x}_2^2$. 4) Find the connection between $\tilde{x}$and $x$ by $\mathbf{x}=\mathbf{Q}\tilde{\mathbf{x}}$ where **Q** is the eigenvector matrix from **A.**
**Tensors:**
$\mathbf{A_{ij}=u\otimes v=uv^T=u_iv_j}$
$\mathbf{A_{ijkl}=u\otimes v\otimes w\otimes x=u_iv_jw_kx_l}$
$\mathbf{I_1(A)=A_{ii}}=\mathrm{trace}(\mathbf{A})=\text{1st Invariant}$
$\mathbf{I_2(A)}=\frac{1}{2}(\mathbf{A_{ii}A_{jj}-A_{ji}A_{ij}})=\mathrm{trace}(\mathbf{A})=\text{2nd Invariant}$
1st invariant of stress is hydrostatic pressure.
Eigenvalues of stress and strain tensors are principal stresses and strains.
Eigenvectors of stresses and strains are the directions.
$\mathbf{I_3(A)=A_{1i}A_{2j}A_{3k}}\varepsilon_{\mathbf{ijk}}=\mathbf{det(A)}$=3rd Invariant
To find eigenvalues solve for the roots of:
$\boldsymbol{\lambda}^3-\mathbf{I_1}\boldsymbol{\lambda}^2+\mathbf{I_2}\boldsymbol{\lambda}-\mathbf{I_3}=0$

$$\mathbf{E_{ij}}=\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\partial\mathbf{u_i}}{\partial\mathbf{x'_j}}+\frac{\partial\mathbf{u_j}}{\partial\mathbf{x'_i}}+\frac{\partial\mathbf{u_k}}{\partial\mathbf{x'_i}}\frac{\partial\mathbf{u_k}}{\partial\mathbf{x'_j}}\right)$$

= Finite strain tensor (as opposed to the small strain tensor).
Where: $\mathbf{x}'$ is the reference position.
**Fourier Series:**
Even function**:** $b_n=0, f(-x)=f(x), \int_{-A}^{A}f(x)dx=2\int_0^A f(x)dx$
Odd function**:** $a_n=b_n=0, f(-x)=-f(x), \int_{-A}^{A}f(x)dx=0$

Decompose any function into its even and odd parts:
$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=f_e(x)+f_o(x)$

$$\text{FS } f=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)dx,\qquad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx$$

$f$ is $2l$ periodic, e.g. if the period is $2\pi, l=\pi$.
Elementary integral formulas:

$$\int_{-l}^{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}dx\begin{cases}0, m\neq n\\ l, m=n\neq 0\\ 2l, m=n=0\end{cases}$$

$$\int_{-l}^{l}\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx\begin{cases}0, m\neq n\\ l, m=n\neq 0\end{cases}$$

$$\int_{-l}^{l}\cos\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx=0, \text{for all } m,n$$

Integration by parts: $\int x^2\sin x\,dx=uv-\int vdu$, where $v'=\sin x, u=x^2$
Questions:
If we have 4 vectors of Rank 3, are they guaranteed to be LD? It seems they are if we have more vectors than the rank…

**Integral Table:**

$$(59) \int \sin ax dx = -\frac{1}{a}\cos ax + C$$

$$(60) \int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$(61) \int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} + C$$

$$(61a) \int \sin^n ax dx = -\frac{1}{a}\cos ax \;\, {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] + C$$

$$(62) \int \cos ax dx = \frac{1}{a}\sin ax + C$$

$$(63) \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$(64) \int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} + C$$

$$(64a) \int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \;\, {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] + C$$

$$(65) \int \sin x \cos x dx = \frac{1}{2}\sin^2 x + c_1 = -\frac{1}{2}\cos^2 x + c_2 = \frac{1}{4}\cos 2x + c_3$$

$$(65a) \int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C, a \neq b$$

$$(66) \int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + C$$

$$\int \sin^2 a\theta d\theta = \frac{\theta}{2} - \frac{\sin 2a\theta}{4a} + C$$

$$\int \ln x \, dx = x \ln x - x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int e^x dx = e^x + C$$

Power rule: $\frac{d}{dx} u^x = u^x \ln \mathrm{u}$

$$\frac{ds}{dt} = \lim_{h \to 0} \frac{\|\mathbf{r}(\mathrm{t}+\mathrm{h}) - \mathbf{r}(\mathrm{t})\|}{h}$$

$$\int_0^\tau \sqrt{1+4t^2} dt = \frac{1}{2}\tau\sqrt{1+4\tau^2} + \frac{1}{4}\ln(2\tau + \sqrt{1+4\tau^2})$$

$$\int \frac{A}{B-x} dx = -A \ln(B - x) + C$$

u substitution for simplifying integrals: 1) Substitute a single variable ($u$) for a hard-to-integrate portion ($x$). 2) Find $du$. 3) Rework limits of integration.